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CLOSED SELF-SHRINKING SURFACES IN \mathbb{R}^3 VIA THE TORUS

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ABSTRACT. We construct many closed, embedded mean curvature self-shrinking surfaces $\Sigma_g^2 \subseteq \mathbb{R}^3$ of high genus $g = 2k$, $k \in \mathbb{N}$.

Each of these shrinking solitons has isometry group equal to the dihedral group on $2g$ elements, and comes from the "gluing", i.e. desingularizing of the singular union, of the two known closed embedded self-shrinkers in \mathbb{R}^3 : The round 2-sphere \mathbb{S}^2 , and Angenent's self-shrinking 2-torus \mathbb{T}^2 of revolution. This uses the results and methods N. Kapouleas developed for minimal surfaces in [Ka97]–[Ka].

1. INTRODUCTION

Recall that a smooth surface $\Sigma^2 \subseteq \mathbb{R}^3$ is a mean curvature self-shrinker if it satisfies the corresponding nonlinear elliptic self-shrinking soliton PDE:

$$(1.1) \quad H(\Sigma) = -\frac{\langle X, \nu_\Sigma \rangle}{2}, \quad X \in \Sigma^2 \subseteq \mathbb{R}^3,$$

where H denotes the mean curvature, X the position vector and ν_Σ is a unit length vector field normal to $\Sigma^2 \subseteq \mathbb{R}^3$.

While important as singularity models in mean curvature flow (see e.g. [Hu90]–[Hu93] and [CM1]–[CM3]), the list of known closed, embedded surfaces satisfying Equation (1.1) is short:

- The round 2-sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$,
- Angenent's (non-circular) 2-torus, in [An92].

Apart from these, there is numerical evidence for the existence of a self-shrinking "fattened wire cube" in [Ch94].

There is a rigorous construction of closed, embedded, smooth mean curvature self-shrinkers with high genus g , embedded in Euclidean space \mathbb{R}^3 . The theorem is the following:

Theorem 1. *For every large enough even integer $g = 2k$, $k \in \mathbb{N}$, there exists a compact, embedded, orientable, smooth surface without boundary $\Sigma_g^2 \subseteq \mathbb{R}^3$, with the properties:*

- (i) Σ_g is a mean curvature self-shrinker of genus g .
- (ii) Σ_g is invariant under the dihedral symmetry group with $2g = 4k$ elements.
- (iii) The sequence $\{\Sigma_g\}$ converges in Hausdorff sense to the union $\mathbb{S}^2 \cup \mathbb{T}^2$, where \mathbb{T}^2 is a rotationally symmetric self-shrinking torus in \mathbb{R}^3 . The convergence is locally smooth away from the two intersection circles constituting $\mathbb{S}^2 \cap \mathbb{T}^2$.

Key words and phrases. Mean curvature flow, self-shrinkers, self-similarity, solitons, minimal surfaces, gluing constructions, stability theory, geodesics, spectral theory.

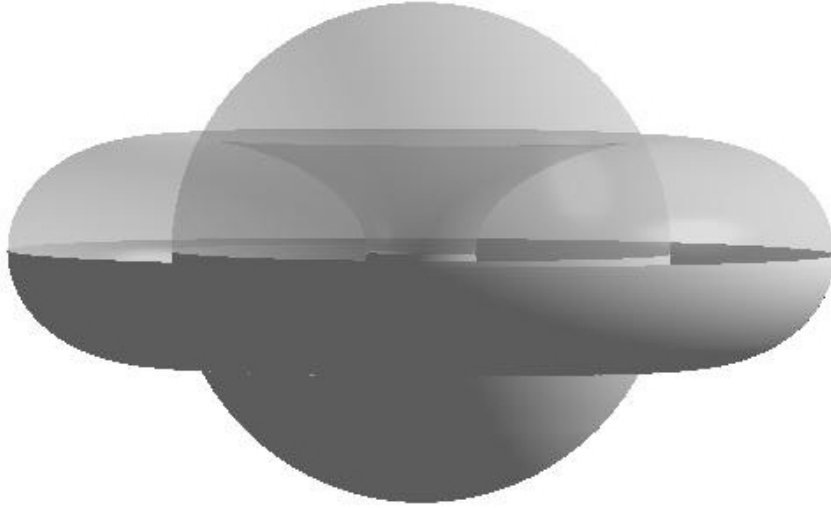


FIGURE 1. The closed, embedded self-similar "toruspheres" $\Sigma_g^2 \subseteq \mathbb{R}^3$ of genus g in Theorem 1. *Showing:* Immersed singular configuration, before handle insertion along two intersection curves. [MATLAB]

This paper consists of: (1) Brief account of the construction and its important components, and (2) Proofs of the central explicit estimates of functions that for some small $\delta, \varepsilon > 0$ in an appropriate sense are δ -close to being eigenfunctions of the stability operator (corresp. δ -Jacobi fields), on surfaces that are ε -close to being self-shrinkers (corresp. ε -geodesics) near a candidate for the self-shrinking torus.

The implications of such estimates are: The existence of a self-shrinking torus \mathbb{T}^2 (via existence of a smooth closed geodesic loop) with useful quantitative estimates of its geometry. Hence it gives conclusions about, the Dirichlet and Neumann problems of the stability operator \mathcal{L} on this "quantitative torus", leading to the main technical result below in Theorem 2.7 which is sufficient to prove Theorem 1.

Lengthier accounts with a thorough treatment of the background and details relating to this problem, and the construction as developed for general compact minimal hypersurfaces in 3-manifolds by Nikolaos Kapouleas in [Ka97]-[Ka05] and presented in detail recently in [Ka11] which, caveat lector, will not be described here, can be found (in the highly symmetric special case) in [Mø]. Note that also the references [Ng06]-[Ng07], and more recently [KKMø] and [Ng11], were concerned with gluing problems for (non-compact) self-shrinkers.

Self-shrinkers are minimal surfaces in Euclidean space with respect to a conformally changed Gaussian metric g :

$$(1.2) \quad \begin{aligned} \Sigma^n \subseteq \mathbb{R}^{n+1} \text{ is a self-shrinker} &\Leftrightarrow H_{g_{ij}}(\Sigma) = 0, \\ g_{ij} &= \frac{\delta_{ij}}{\exp(|X|^2/2n)}, \quad X \in \mathbb{R}^{n+1}. \end{aligned}$$

Recall the constructions by Nicos Kapouleas (in [Ka97]-[Ka11]), concerning desingularization of a finite collection of compact minimal surfaces in a general ambient Riemannian 3-manifold (M^3, g) . The conditions for the construction to work, in our situation, are the following where the collection is identified with one immersed surface \mathcal{W} with intersections along the (smooth) curve $\underline{\mathcal{C}}$, which can have several connected components.

Conditions 1.1 ([Ka05]-[Ka11]).

- (I) *There are no points of triple intersection, all intersections are transverse and $\underline{\mathcal{C}} \cap \partial\mathcal{W} = \emptyset$ holds.*
- (N1) *The kernel for the linearized operator*

$$(1.3) \quad \mathcal{L} = \Delta + |A|^2 + \text{Ric}(\nu, \nu) \quad \text{on} \quad \mathcal{W},$$
with Dirichlet conditions on $\partial\mathcal{W}$, is trivial (unbalancing condition).
- (N2) *The kernel for the linearized operator \mathcal{L} on $\hat{\mathcal{W}}$, with Dirichlet conditions on $\partial\hat{\mathcal{W}}$, is trivial (flexibility condition).*

Instead of (N1) one may substitute:

- (N1') *The kernel for the linearized operator \mathcal{L} on \mathcal{W} , with Neumann conditions on $\partial\mathcal{W}$, is trivial (unbalancing condition).*

The Neumann version (N1') of the non-degeneracy conditions will be used for the construction of the closed, embedded self-shrinkers where one solves the self-shrinker equations for graphs with the Neumann conditions over the circle of intersection of the torus and symmetry plane $\mathbb{T}^2 \cap \mathcal{P}$, where \mathbb{T}^2 denotes a self-shrinking torus. Then the closed surfaces are obtained via doubling them by reflection through this plane.

An important version of the above conditions, is the one obtained by imposing symmetries, say under a group G , throughout the construction. Indeed, one may then restrict to verifying the non-degeneracy conditions (N1)-(N2) under the additional assumption of the symmetries in G .

While some properties of the Jacobi fields can be deduced from the known eigenvalues and -functions for the stability operator \mathcal{L} (see e.g. [CM2]) one does not obtain enough accurate information for our purposes. We show how to estimate the quantities using the (generalized) Bellman-Grönwall's inequalities for second order Sturm-Liouville problems, and explicit test functions.

The care one needs to exercise when estimating the explicit constants, as well as the number and complexity of the barriers and test functions one needs to choose, becomes non-trivial owing to mainly two factors: 1) The large Lipschitz constants of the PDE system (relative to the scale of the self-shrinking torus). The geometric reason this happens is that the Gauß curvature in the metric on Angenent's upper half-plane has a maximal value of around 30 along the candidate torus (with the maximum occurring at the point nearest to the origin in \mathbb{R}^3), giving naive characteristic conjugate distances of down to $\frac{\pi}{\sqrt{K}} \sim 0.5$, while the circumference of the torus is around ~ 7 , in the metric. Hence the solutions to Jacobi's equation can be expected to, and indeed does, oscillate several times around the circumference of \mathbb{T}^2 , in a non-uniform way and yet fail to "match up". Furthermore: 2) The location of

the conjugate points on \mathbb{S}^2 for the appropriate Jacobi equation is furthermore very near to the singular curve $\underline{\mathcal{C}}$ (i.e. the boundaries of the connected components of $\mathcal{M} \setminus \underline{\mathcal{C}}$), which requires also requires tighter estimates.

Hence it takes work to strengthen the estimates to a useful form. One device to do this is what we call the "sesqui"-shooting problem in Definition 2.4, for identifying the position of the torus, where "sesqui" refers to the fact that it is a double shooting problem but with a compatibility condition linking the two: That each pair of curves always meet at a simple, explicitly known solution. Here we take the round cylinder of radius $\sqrt{2}$ as reference. Since the errors are exponential in the integral of the Lipschitz constants, one may by virtue of the sesqui-shooting roughly take the square-root of the errors, which allows us to obtain bounds with explicit constants of the order of 10^1 – 10^2 instead of 10^4 .

Listing our explicit choices of test functions (f.ex. adequate piecewise polynomial choices can easily be found using Taylor expansions at a few selected points) to use with the key estimates in Section 2 is likely not in itself very enlightening for the reader at this point, and with further simplification of such yet to be worked out, we postpone describing them in detail, being confident to later be able to supply a collection for which the properties can be checked manually with minimal time consumption.

2. ε -GEODESICS, δ -JACOBI FIELDS AND A SELF-SHRINKING TORUS \mathbb{T}^2

2.1. Existence of \mathbb{T}^2 , with geometric estimates. In order to later understand precisely the kernel of the stability operators, we need to establish a detailed quantitative version of the existence of Angenent's torus. Towards the end of this section we will arrive at the basic, explicit estimates on the location and geometric quantities of such a torus, but we will first work out the explicit estimates for also the Jacobi equation before finalizing the choices in the estimate, in order to require as little as possible of the test functions.

We note that much less complicated estimates would ensure the mere existence, leading to a different proof of the self-shrinking "doughnut" existence result from [An92], in the 3-dimensional case. But as explained, here we will need very precise estimates of several aspects of the geometry of \mathbb{T}^2 .

Proposition 2.1. *Let $\varepsilon_{gap} = 10^{-3}$. There exists a closed, embedded, self-shrinking torus of revolution \mathbb{T}^2 with the following properties:*

- (1) *The torus is $(x_1 \mapsto -x_1)$ -symmetric.*
- (2) *\mathbb{T}^2 intersects the x_2 -axis orthogonally at two heights $a^+ > a^- > 0$,*

$$(2.1) \quad \frac{4034}{1217} - \frac{5}{2}\varepsilon_{gap} < a^+ < \frac{4034}{1217} + \frac{5}{2}\varepsilon_{gap},$$

$$(2.2) \quad \frac{7}{16} - \frac{3}{98} < a^- < \frac{7}{16} + \frac{3}{98}.$$

- (3) *\mathbb{T}^2 intersects the sphere \mathbb{S}^2 of radius 2 at two points $p^\pm = (\pm x_{\mathbb{S}^2}, y_{\mathbb{S}^2})$, where:*

$$(2.3) \quad \|p^\pm - (\pm \frac{29}{32}, \frac{41}{23})\|_{\mathbb{R}^2} \leq 5\varepsilon_{gap}.$$

Correspondingly, the angles $\angle(\pm \vec{e}_1, p^\pm)$ from the x_1 -axis satisfy:

$$(2.4) \quad |\angle(\pm \vec{e}_1, p^\pm) - \frac{11}{10}| \leq 5\varepsilon_{\text{gap}}.$$

Remark 2.2. Note that we do not assert or prove uniqueness of \mathbb{T}^2 with these properties, even though this is expected to be true.

Proof. We first need to describe the double shooting problem, or rather "sesqui"-shooting problem since we remove one parameter. For this we need an elementary lemma, which can either be proved (in a weaker version) using the approximation methods described later in this section, or by analysis directly of the ODE in (2.32).

Lemma 2.3. For each $d \in [0, \sqrt{2}]$ let $\gamma_d : [0, \infty) \rightarrow \mathbb{R}^+ \times \mathbb{R}$ be the geodesic starting at $(0, d)$ with initial derivative $\gamma'_d(0) = (1, 0)$ and consider the first time $t^0(d)$ such that γ_d intersects the cylinder geodesic $\{x_2 = \sqrt{2}\}$.

Then the function $I : [0, \sqrt{2}] \rightarrow [0, \sqrt{2}]$ given by

$$I(d) := x_1(\gamma_s(t^0(d))),$$

is continuous and strictly increasing.

Definition 2.4 ("Sesqui"-Shooting Problem).

- (i) Fix $a > \sqrt{2}$.
- (ii) Let $\gamma_a : [0, t_a] \rightarrow \mathbb{R}^+ \times \mathbb{R}$ be the fully extended geodesic contained in $\{x_1 \geq 0\}$, with initial conditions $\gamma_a(0) = (0, a)$ and $\gamma'_a(0) = (1, 0)$.
- (iii) By the maximum principle for (2.32), we have $\gamma_a \cap \{x_2 = \sqrt{2}\} \neq \emptyset$. Assume that the first point of γ_a crossing $\sqrt{2}$ belongs to $\{0 < x_1 \leq \sqrt{2}\}$, and denote the time it happens by $t^0(a)$.
- (iv) By the preceding Lemma, there exists a unique $b(a) \in [0, \sqrt{2}]$ and $s^0(a)$ such that for γ_b on $[0, s^0]$ with $\gamma_b(0) = (0, b)$ and $\gamma'_b(0) = (1, 0)$, we have

$$\gamma_a(t^0(a)) = \gamma_b(s^0(a)).$$

- (iv) Define $\Phi(a)$ by

$$\Phi(a) := \frac{\vec{e}_3 \cdot (\gamma'_a(t^0(a)) \times \gamma'_b(s^0(a)))}{|\gamma'_a(t^0(a))| |\gamma'_b(s^0(a))|},$$

or equivalently the oriented angle between the tangents of γ_a and γ_b at the intersection point.

The following lemma is clear from the definition:

Lemma 2.5. The function Φ in Definition 2.4 is well-defined and continuous, on the open connected set of values $a > \sqrt{2}$ with the property that the first point of intersection of γ_a with $\{x_2 = \sqrt{2}\}$ belongs to $(0, \sqrt{2})$.

The strategy for the proof of the proposition will now be to prove for two different nearby pairs (a^+, b^+) and (a^-, b^-) , that will be chosen such that their geodesics intersect on the cylinder $\{x_2 = \sqrt{2}\}$, that:

$$(2.5) \quad \Phi(a^+) > 0,$$

$$(2.6) \quad \Phi(a^-) < 0.$$

Existence then follows, from the intermediate value theorem, of a pair (a^0, b^0) such that:

$$(2.7) \quad \Phi(a^0) = 0,$$

$$(2.8) \quad a^- < a^0 < a^+,$$

$$(2.9) \quad b^- < b^0 < b^+.$$

The estimates leading to the proof of (2.5)–(2.6) will then lead to the estimates in the proposition.

Consider now a curve (parametrized by any parameter t),

$$\gamma(t) = (x(t), y(t)).$$

Then the equation for γ to generate a self-shrinker by rotation reads:

$$x''y' - y''x' = \left[\frac{yx' - xy'}{2} - \frac{x'}{y} \right] ((x')^2 + (y')^2).$$

Recall that if we define the operator \mathcal{M}_1 acting on C^2 -functions $u : I \rightarrow \mathbb{R}$, for some interval $I \subseteq [0, \frac{1}{2}]$, by

$$(2.10) \quad \mathcal{M}_1(u, p) := \left[\frac{x_1 p - u}{2} + \frac{1}{u} \right] (1 + p^2),$$

then:

Lemma 2.6. *The function $u(x_1)$, a graph over the x_1 -axis, generates a self-shrinker by rotation (around x_1) if and only if*

$$(2.11) \quad u'' = \mathcal{M}_1(u, u').$$

Likewise for graphs over the x_2 -axis, defining

$$(2.12) \quad \mathcal{M}_2(f, q) := \left[\left(\frac{x_2}{2} - \frac{1}{x_2} \right) q - \frac{f}{2} \right] (1 + q^2),$$

again characterizes such solutions via $f'' = \mathcal{M}_2(f, f')$.

We compute that

$$(2.13) \quad \frac{\partial}{\partial u} \mathcal{M}_1(u, p) = \left[-\frac{1}{2} - \frac{1}{u^2} \right] (1 + p^2),$$

$$(2.14) \quad \frac{\partial}{\partial p} \mathcal{M}_1(u, p) = \frac{x_1}{2} (1 + p^2) + 2p \left[\frac{x_1 p - u}{2} + \frac{1}{u} \right],$$

$$(2.15) \quad \frac{\partial}{\partial f} \mathcal{M}_2(f, q) = -\frac{1}{2} (1 + q^2),$$

$$(2.16) \quad \frac{\partial}{\partial q} \mathcal{M}_2(f, q) = \left(\frac{x_2}{2} - \frac{1}{x_2} \right) (1 + 3q^2) - fq.$$

We will first consider graphs $u : [0, \frac{3}{5}] \rightarrow \mathbb{R}$. Now, consider an approximate solution U , and we fix the quantity ε_1^T , later to be chosen, which reflects the order of magnitude of the precision we wish to determine the position of \mathbb{T}^2 with on this interval.

$$(2.17) \quad \mathcal{M}_1(U, U') - \mathcal{M}_1(u, u') = (U - u) \frac{\partial}{\partial u} \mathcal{M}_1(\xi, u') + (U' - u') \frac{\partial}{\partial p} \mathcal{M}_1(U, \xi'),$$

for some $\xi \in [U(x_1), u(x_1)]$ and $\xi' \in [U'(x_1), u'(x_1)]$.

Top of the torus: x_1 -graph

Assume we have the uniform estimates:

$$|U'' - \mathcal{M}_1(U, U')| \leq \varepsilon_1^T,$$

for some ε_1^T . Assume also, for the argument (and later show this to be self-consistent, i.e. propagated by the equation), the following bounds:

$$\begin{aligned} \int_0^{x_1} \left| \frac{1}{2} + \frac{1}{\xi^2} \right| (1 + (u'(x_1))^2) dx_1 &\leq \frac{4}{5} x_1, \quad x_1 \in [0, \frac{3}{5}], \quad |\xi - U(x_1)| \leq \varepsilon_0, \\ \int_0^{x_1} \left| \frac{x_1}{2} (1 + (\xi')^2) + 2\xi' \left[\frac{x_1 \xi' - U(x_1)}{2} + \frac{1}{U(x_1)} \right] \right| dx_1 &\leq \frac{8}{3} x_1^2, \quad x_1 \in [0, \frac{3}{5}], \quad |\xi' - U'(x_1)| \leq \varepsilon_0. \end{aligned}$$

If we let $\varphi(x_1) := |U'(x_1) - u'(x_1)|$, we can now estimate (recall that $||f'|' = |f''|$ almost everywhere, as an easy consequence of Sard's Theorem):

$$\begin{aligned} (2.18) \quad \varphi'(x_1) &\leq ||U' - u'|' \stackrel{\text{a.e.}}{=} |U'' - u''| \\ &\leq |\mathcal{M}_1(U, U') - \mathcal{M}_1(u, u')| + \varepsilon_1^T \\ &\leq |\partial_u \mathcal{M}_1(\xi, u')| |U - u| + |\partial_p \mathcal{M}_1(U, \xi')| |U' - u'| + \varepsilon_1^T \\ &\leq |\partial_p \mathcal{M}_1(U, \xi')| \varphi(x_1) + |\partial_u \mathcal{M}_1(\xi, u')| \int_0^{x_1} \varphi(s) ds + |\partial_u \mathcal{M}_1(\xi, u')| |U(0) - u(0)| + \varepsilon_1^T. \end{aligned}$$

Thus we integrate this inequality, which holds almost everywhere with respect to the Lebesgue measure, and get:

$$\begin{aligned} \varphi(x_1) &\leq \varphi(0) + \int_0^{x_1} |\partial_p \mathcal{M}_1(U, \xi')(s)| \varphi(s) ds + \int_0^{x_1} |\partial_u \mathcal{M}_1(\xi, u')(t)| \int_0^t \varphi(s) ds dt \\ &\quad + \frac{4}{5} x_1 |U(0) - u(0)| + \varepsilon_1^T x_1 \\ &\leq \varphi(0) + \frac{4}{5} x_1 |U(0) - u(0)| + \varepsilon_1^T x_1 + \int_0^{x_1} \left[|\partial_p \mathcal{M}_1(U, \xi')(s)| + \frac{4}{5} x_1 \right] \varphi(s) ds. \end{aligned}$$

We are now ready to use the integral form of Grönwall-Bellman's inequality, namely for $\alpha(t)$ a non-decreasing function:

$$\forall t \in I : \varphi(t) \leq \alpha(t) + \int_a^t \beta(s) \varphi(s) ds \implies \forall t \in I : \varphi(t) \leq \alpha(t) \exp \left\{ \int_a^t \beta(s) ds \right\}.$$

Here we thus conclude from the above, that

$$(2.19) \quad \varphi(x_1) \leq \left[\varphi(0) + \frac{4}{5} x_1 |U(0) - u(0)| + \varepsilon_1^T x_1 \right] \exp \left\{ \frac{8}{3} x_1^2 + \frac{4}{5} x_1^2 \right\},$$

so that here we obtain the estimates (for $\varphi(0) = 0$):

$$\begin{aligned}
|U'(x_1) - u'(x_1)| &\leq \frac{3}{5} \exp(\frac{156}{125})(\varepsilon_1^T + \frac{4}{5}|U(0) - u(0)|), \\
|U(x_1) - u(x_1)| &\leq |U(0) - u(0)| + (\varepsilon_1^T + \frac{4}{5}|U(0) - u(0)|) \int_0^{\frac{3}{5}} s \exp(\frac{52}{15}s^2) ds \\
&= |U(0) - u(0)| + \frac{15}{104} (\exp(\frac{52}{15}(\frac{3}{5})^2) - 1)(\varepsilon_1^T + \frac{4}{5}|U(0) - u(0)|) \\
&\leq \frac{23}{80}|U(0) - u(0)| + \frac{9}{25}\varepsilon_1^T.
\end{aligned}$$

Top of the torus: x_2 -graph to the sphere

We continue with the next part, which is graphical over the x_2 -axis. In this region we again let $\psi(x_2) := |F' - f'|$. Here we will assume the estimate

$$(2.20) \quad |F'' - \mathcal{M}_2(F, F')| \leq \varepsilon_2^T,$$

and furthermore the (the second one to later be proven consistent) estimates on $x_2 \in [y_{\mathbb{S}^2}, u_T(3/5)]$

$$\begin{aligned}
I_u^{(2)} &= \int_{x_2}^{u_T(3/5)} \left| \frac{1}{2}(1 + (F')^2) \right| dx_2 \leq \frac{13}{8} - \frac{x_2}{2}, \\
I_p^{(2)} &= \int_{x_2}^{u_T(3/5)} \left| \left(\frac{x_2}{2} - \frac{1}{x_2} \right) (1 + 3(\xi')^2) - F(x_2)\xi' \right| dx_2 \leq \frac{7}{4} - \frac{9}{10}(x_2 - y_{\mathbb{S}^2})^2.
\end{aligned}$$

We can then estimate as follows:

$$\begin{aligned}
\psi'(x_2) &\leq |\partial_u \mathcal{M}_2| |F - f| + |\partial_p \mathcal{M}_2| |F' - f'| + \varepsilon_2^T \\
&\leq |\partial_u \mathcal{M}_2| \int_{x_2}^a \psi(s) ds + |\partial_p \mathcal{M}_2| \psi(x_2) + |\partial_u \mathcal{M}_2| |F(a) - f(a)| + \varepsilon_2^T,
\end{aligned}$$

which integrates to

$$\begin{aligned}
(2.21) \quad \psi(x_2) &\leq \int_{x_2}^{u_T(3/5)} [|\partial_p \mathcal{M}_2| + \frac{13}{8} - \frac{x_2}{2}] \psi(s) ds + (\frac{13}{8} - \frac{x_2}{2}) |F(u_T(3/5)) - f(u_T(3/5))| \\
(2.22) \quad &+ \varepsilon_2^T (u_T(3/5) - x_2) + \psi(a),
\end{aligned}$$

so that, again by Grönwall-Bellman,

$$\begin{aligned}
\psi(x_2) &\leq \left[(\frac{13}{8} - \frac{x_2}{2}) |F(a) - f(a)| + |F'(a) - f'(a)| + \varepsilon_2^T (u_T(3/5) - x_2) \right] \times \\
&\quad \exp \left\{ \frac{7}{4} - \frac{9}{10}(x_2 - y_{\mathbb{S}^2})^2 + (\frac{13}{8} - \frac{x_2}{2})(u_T(3/5) - x_2) \right\}.
\end{aligned}$$

The endpoint estimates are:

$$\begin{aligned}
\psi(y_{\mathbb{S}^2}) &\leq \frac{59}{4} |F(u_T(3/5)) - f(u_T(3/5))| + \frac{54}{5} |F'(u_T(3/5)) - f'(u_T(3/5))| + \frac{377}{20} \varepsilon_2^T \\
&\leq \frac{59}{4} \frac{|U(F(u_T(3/5))) - u(f(u_T(3/5)))|}{|U'(F(\xi))|} + \frac{54}{5} \frac{|U'(F(u_T(3/5))) - u'(f(u_T(3/5)))|}{|U'(F(u_T(3/5)))||u'(f(u_T(3/5)))|} + \frac{377}{20} \varepsilon_2^T \\
&\leq \frac{59}{4} \frac{10}{11} (\frac{9}{25} \varepsilon_1^T + \frac{23}{80} |U(0) - u(0)|) + \frac{54}{5} (\frac{10}{11})^2 \frac{3}{5} \exp(\frac{156}{125})(\varepsilon_1^T + \frac{4}{5} |U(0) - u(0)|) + \frac{377}{20} \varepsilon_2^T \\
&\leq 27 |U(0) - u(0)| + 34 \varepsilon_1^T + 19 \varepsilon_2^T.
\end{aligned}$$

Integrating the above estimate for $\psi(x_2)$, we also get:

$$\begin{aligned} |F(y_{\mathbb{S}^2}) - f(y_{\mathbb{S}^2})| &\leq (1 + \frac{63}{8})|F(u_T(3/5)) - f(u_T(3/5))| + \frac{83}{20}|F'(u_T(3/5)) - f'(u_T(3/5))| + \frac{137}{20}\varepsilon_2^T \\ &\leq 12|U(0) - u(0)| + 15\varepsilon_1^T + 7\varepsilon_2^T. \end{aligned}$$

Top of the torus, x_2 -graph from sphere to cylinder

We consider the region between the sphere and cylinder, and let again $\psi(x_2) := |F' - f'|$. Here we will assume the estimate

$$(2.23) \quad |F'' - \mathcal{M}_2(F, F')| \leq \varepsilon_3^T,$$

and furthermore the estimates on $x_2 \in [y_{\mathbb{S}^2}, \sqrt{2}]$

$$\begin{aligned} I_u^{(2)} &= \int_{x_2}^{y_{\mathbb{S}^2}} \left| \frac{1}{2}(1 + (F')^2) \right| dx_2 \leq \frac{1}{4}(y_{\mathbb{S}^2} - x_2), \\ I_p^{(2)} &= \int_{x_2}^{y_{\mathbb{S}^2}} \left| \left(\frac{x_2}{2} - \frac{1}{x_2} \right) (1 + 3(\xi')^2) - F(x_2)\xi' \right| dx_2 \leq \frac{27}{50}(y_{\mathbb{S}^2} - x_2), \end{aligned}$$

which again should later be checked hold in the regime we consider.

We estimate as follows:

$$\begin{aligned} \psi'(x_2) &\leq |\partial_u \mathcal{M}_2| |F - f| + |\partial_p \mathcal{M}_2| |F' - f'| + \varepsilon_2^T \\ &\leq |\partial_u \mathcal{M}_2| \int_{x_2}^{y_{\mathbb{S}^2}} \psi(s) ds + |\partial_p \mathcal{M}_2| \psi(x_2) + |\partial_u \mathcal{M}_2| |F(y_{\mathbb{S}^2}) - f(y_{\mathbb{S}^2})| + \varepsilon_3^T, \end{aligned}$$

which integrates to:

$$(2.24) \quad \psi(x_2) \leq \psi(y_{\mathbb{S}^2}) + \int_{x_2}^{y_{\mathbb{S}^2}} [|\partial_p \mathcal{M}_2| + \frac{1}{4}(y_{\mathbb{S}^2} - x_2)] \psi(s) ds + \frac{1}{4}(y_{\mathbb{S}^2} - x_2) |F(y_{\mathbb{S}^2}) - f(y_{\mathbb{S}^2})| + \varepsilon_3^T (y_{\mathbb{S}^2} - x_2),$$

so that, again by Grönwall-Bellman,

$$\begin{aligned} \psi(x_2) &\leq \left[\frac{1}{4}(y_{\mathbb{S}^2} - x_2) |F(y_{\mathbb{S}^2}) - f(y_{\mathbb{S}^2})| + |F'(y_{\mathbb{S}^2}) - f'(y_{\mathbb{S}^2})| + \varepsilon_3^T (y_{\mathbb{S}^2} - x_2) \right] \times \\ &\quad \exp \left\{ \frac{27}{50}(y_{\mathbb{S}^2} - x_2) + \frac{1}{4}(y_{\mathbb{S}^2} - x_2)^2 \right\}. \end{aligned}$$

Inserting the previous estimate gives:

$$\begin{aligned} \psi(\sqrt{2}) &\leq \frac{1}{8}|F(y_{\mathbb{S}^2}) - f(y_{\mathbb{S}^2})| + \frac{4}{3}|F'(y_{\mathbb{S}^2}) - f'(y_{\mathbb{S}^2})| + \frac{1}{2}\varepsilon_3^T \\ &\leq \frac{1}{8}(12|U(0) - u(0)| + 15\varepsilon_1^T + 7\varepsilon_2^T) + \frac{4}{3}(27|U(0) - u(0)| + 34\varepsilon_1^T + 19\varepsilon_2^T) + \frac{1}{2}\varepsilon_3^T. \end{aligned}$$

Hence, we finally obtain the estimate:

$$|F'(\sqrt{2}) - f'(\sqrt{2})| \leq 36|U(0) - u(0)| + 45\varepsilon_1^T + 25\varepsilon_2^T + \frac{1}{2}\varepsilon_3^T.$$

Integrating the estimates, we also get:

$$\begin{aligned} |F(y_{\mathbb{S}^2}) - f(y_{\mathbb{S}^2})| &\leq (1 + \frac{1}{50})|F(y_{\mathbb{S}^2}) - f(y_{\mathbb{S}^2})| + \frac{33}{80}|F'(y_{\mathbb{S}^2}) - f'(y_{\mathbb{S}^2})| + \frac{2}{25}\varepsilon_3^T \\ &\leq 24|U(0) - u(0)| + 30\varepsilon_1^T + 15\varepsilon_2^T + \frac{2}{25}\varepsilon_3^T. \end{aligned}$$

Bottom of the torus: x_1 -graph from the plane

Assume uniform estimates:

$$|U'' - \mathcal{M}_1(U, U')| \leq \varepsilon_1^B,$$

for some ε_1^B . Assume also, for the argument (and later show this to be self-consistent, i.e. propagated by the equation), the following bounds (for $x_1 \in [0, \frac{1}{2}]$):

$$\begin{aligned} \int_0^{x_1} \left| \frac{1}{2} + \frac{1}{\xi^2} \right| (1 + (u'(x_1))^2) dx_1 &\leq \frac{29}{5}x_1 + \frac{1}{20}, \quad |\xi - U(x_1)| \leq \varepsilon_0, \\ \int_0^{x_1} \left| \frac{x_1}{2} (1 + (\xi')^2) + 2\xi' \left[\frac{x_1\xi' - U(x_1)}{2} + \frac{1}{U(x_1)} \right] \right| dx_1 &\leq 4x_1^2 + \frac{1}{5}x_1, \quad |\xi' - U'(x_1)| \leq \varepsilon_0. \end{aligned}$$

As always, we get with $\varphi(x) = |U'(x) - u'(x)|$:

$$\begin{aligned} \varphi(x_1) &\leq \varphi(0) + \int_0^{x_1} |\partial_p \mathcal{M}_1(U, \xi')(s)| \varphi(s) ds + \int_0^{x_1} |\partial_u \mathcal{M}_1(\xi, u')(t)| \int_0^t \varphi(s) ds dt \\ &\quad + \left(\frac{29}{5}x_1 + \frac{1}{20} \right) |U(0) - u(0)| + \varepsilon_1^B x_1 \\ &\leq \varphi(0) + \left(\frac{29}{5}x_1 + \frac{1}{20} \right) |U(0) - u(0)| + \varepsilon_1^B x_1 + \int_0^{x_1} \left[|\partial_p \mathcal{M}_1(U, \xi')(s)| + \frac{29}{5}x_1 + \frac{1}{20} \right] \varphi(s) ds. \end{aligned}$$

Using once again Grönwall-Bellman's inequality,

$$(2.25) \quad \varphi(x_1) \leq \left[\varphi(0) + \left(\frac{29}{5}x_1 + \frac{1}{20} \right) |U(0) - u(0)| + \varepsilon_1^B x_1 \right] \exp \left\{ \frac{29}{5}x_1^2 + \frac{x_1}{20} + 4x_1^2 + \frac{1}{5}x_1 \right\},$$

and we obtain the estimates (for $\varphi(0) = 0$):

$$\begin{aligned} |U'(x_1) - u'(x_1)| &\leq \left[\left(\frac{29}{5}x_1 + \frac{1}{20} \right) |U(0) - u(0)| + \varepsilon_1^B x_1 \right] \exp \left\{ \frac{49}{5}x_1^2 + \frac{x_1}{4} \right\} \\ |U'(\tfrac{1}{2}) - u'(\tfrac{1}{2})| &\leq 39|U(0) - u(0)| + \frac{28}{5}\varepsilon_1^B, \\ |U(x_1) - u(x_1)| &\leq |U(0) - u(0)| \left(1 + \int_0^{x_1} \left(\frac{29}{5}s + \frac{1}{20} \right) \exp \left\{ \frac{49}{5}s^2 + \frac{s}{4} \right\} ds \right) \\ &\quad + \varepsilon_1^B \int_0^{x_1} s \exp \left\{ \frac{49}{5}s^2 + \frac{s}{4} \right\} ds \\ |U(\tfrac{1}{2}) - u(\tfrac{1}{2})| &\leq \frac{23}{5}|U(0) - u(0)| + \frac{3}{5}\varepsilon_1^B. \end{aligned}$$

Bottom: x_2 -graph to cylinder

We continue with the next part, which is graphical over the x_2 -axis. In this region we again let $\psi(x_2) = \psi_B(x_2) := |F' - f'|$. Here we will assume the estimate

$$(2.26) \quad |F'' - \mathcal{M}_2(F, F')| \leq \varepsilon_2^B,$$

and furthermore the (to later be proven consistent) estimates on $x_2 \in [a_0, \sqrt{2}]$ (where $|a_0 - \frac{28}{39}| \leq \varepsilon_0$).

$$\begin{aligned} I_u^{(2)} &= \int_{a_0}^{x_2} \left| \frac{1}{2} (1 + (F')^2) \right| dx_2 \leq \frac{16}{25}x_2 - \frac{3}{7}, \\ I_p^{(2)} &= \int_{a_0}^{x_2} \left| \left(\frac{x_2}{2} - \frac{1}{x_2} \right) (1 + 3(\xi')^2) - F(x_2)\xi' \right| dx_2 \leq \frac{9}{10} - \frac{11}{10} \left(x_2 - \frac{3}{2} \right)^2. \end{aligned}$$

We can then estimate as follows:

$$\begin{aligned}\psi'(x_2) &\leq |\partial_u \mathcal{M}_2| |F - f| + |\partial_p \mathcal{M}_2| |F' - f'| + \varepsilon_2^B \\ &\leq |\partial_u \mathcal{M}_2| \int_a^{x_2} \psi(s) ds + |\partial_p \mathcal{M}_2| \psi(x_2) + |\partial_u \mathcal{M}_2| |F(a) - f(a)| + \varepsilon_2^B,\end{aligned}$$

which integrates to

$$(2.27) \quad \psi(x_2) \leq \int_{a_0}^{x_2} [|\partial_p \mathcal{M}_2| + \frac{16}{25}x_2 - \frac{3}{7}] \psi(s) ds + (\frac{16}{25}x_2 - \frac{3}{7})|F(a_0) - f(a_0)| + \varepsilon_2^B(x_2 - a_0) + \psi(a).$$

By Grönwall-Bellman,

$$\begin{aligned}\psi(x_2) &\leq \left[(\frac{16}{25}x_2 - \frac{3}{7})|F(a) - f(a)| + |F'(a) - f'(a)| + \varepsilon_2^B(x_2 - a_0) \right] \times \\ &\quad \exp \left\{ \frac{9}{10} - \frac{11}{10} (x_2 - \frac{3}{2})^2 + (x_2 - a_0) (\frac{16}{25}x_2 - \frac{3}{7}) \right\} \\ &\leq \frac{13}{8} |F(a_0) - f(a_0)| + \frac{17}{5} |F'(a_0) - f'(a_0)| + \frac{69}{29} \varepsilon_2^B \\ &\leq \frac{13}{8} \frac{|U(F(a_0)) - u(f(a_0))|}{|U'(F(\xi))|} + \frac{17}{5} \frac{|U'(F(a_0)) - u'(f(a_0))|}{|U'(F(a_0))||u'(f(a_0))|} + \frac{69}{29} \varepsilon_2^B \\ &\leq \frac{13}{8} \frac{10}{12} (\frac{3}{5} \varepsilon_1^B + \frac{23}{5} |U(0) - u(0)|) + \frac{17}{5} (\frac{10}{12})^2 (\frac{28}{5} \varepsilon_1^B + 39 |U(0) - u(0)|) + \frac{69}{29} \varepsilon_2^B \\ &\leq 100 |U(0) - u(0)| + 15 \varepsilon_1^B + \frac{5}{2} \varepsilon_2^B.\end{aligned}$$

Integrating the first line of this estimate, we also get:

$$\begin{aligned}|F(\sqrt{2}) - f(\sqrt{2})| &\leq (1 + \frac{23}{50}) |F(a_0) - f(a_0)| + \frac{14}{9} |F'(a_0) - f'(a_0)| + \frac{2}{3} \varepsilon_2 \\ &\leq 48 |U(0) - u(0)| + 7 \varepsilon_1^B + \frac{2}{3} \varepsilon_2^B.\end{aligned}$$

Hence we will arrange that for test functions (curves) $\gamma_{\text{up}}, \gamma_{\text{low}} : [0, 1] \rightarrow \mathbb{R} \times \mathbb{R}_+$:

$$(2.28) \quad \gamma_{\text{down}}(0) = (0, \frac{4034}{1217} - \frac{5}{2} \varepsilon_{\text{ud}})$$

$$(2.29) \quad \gamma_{\text{up}}(0) = (0, \frac{4034}{1217} + \frac{5}{2} \varepsilon_{\text{ud}}).$$

$$(2.30)$$

Then, as explained earlier, the intermediate value theorem applied to the sesqui-shooting problem implies the existence of the torus with the estimates (2.1)–(2.2).

To see Property (3), we recall that

$$(2.31) \quad |F(y_{\mathbb{S}^2}) - f(y_{\mathbb{S}^2})| \leq \frac{317}{25} |U(0) - u(0)| + 11 \varepsilon_1 + \frac{27}{5} \varepsilon_2,$$

from which the estimate (2.3) follows. □

2.2. The stability operator \mathcal{L} on subsets of \mathbb{S}^2 and \mathbb{T}^2 . Recall the self-shrinker equation for a smooth oriented surface $S \subseteq \mathbb{R}^3$ to be a self-shrinker (shrinking towards the origin with singular time $T = 1$) is

$$(2.32) \quad H_S(\vec{X}) - \frac{1}{2} \vec{X} \cdot \vec{\nu}_S(\vec{X}) = 0,$$

for each $\vec{X} \in S$, where by convention $H_S = \sum_1^n \kappa_i$ is the sum of the signed principal curvatures w.r.t. the chosen normal $\vec{\nu}_S$ (i.e. $H = 2$ for the sphere with outward pointing $\vec{\nu}$). We have normalized Equation (2.32) so that $T = 1$ is the singular time.

For a smooth normal variation \vec{X}_t determined by a function u via $X_t = X_0 + tu\vec{\nu}_S$, where \vec{X}_0 parametrizes S , the pointwise linear change in (minus) the quantity on the left hand side in (2.32) is given by the stability operator (see the Appendix, and also [CM1]-[CM2] for more properties of this operator)

$$(2.33) \quad \mathcal{L}_S u = \Delta_S u + |A_S|^2 u - \frac{1}{2} \left(\vec{X} \cdot \nabla_S u - u \right).$$

We are now ready to prove the main technical theorem in this paper, concerning the kernel on the connected components (with boundaries) of

$$(\mathbb{S}^2 \cup \mathbb{T}^2) \setminus (\mathbb{S}^2 \cap \mathbb{T}^2),$$

where \mathbb{T}^2 is the accurately estimated "quantitative" torus

Theorem 2.7. *There exists $N > 0$ large enough that for the below six surfaces with boundary S_1, \dots, S_6 ,*

$$\ker \mathcal{L}_{S_i} = \{0\} \quad \text{for } i = 1, \dots, 6 \quad [w/ \text{ indicated boundary conditions}],$$

when imposing at least N -fold rotational symmetry:

- (1) *The surfaces $S_1 = \mathbb{S}^2 \cap \{x_1 \geq 0\}$ and $S_2 = \mathbb{T}^2 \cap \{x_1 \geq 0\}$ [Neumann conditions].*
- (2) *The components S_3 and S_4 of $\{x_1 \geq 0\} \cap \mathbb{S}^2 \setminus (\mathbb{S}^2 \cap \mathbb{T}^2)$ [Neumann conditions on $\{x_1 = 0\}$, Dirichlet conditions elsewhere].*
- (3) *The components S_5 and S_6 of $\{x_1 \geq 0\} \cap \mathbb{T}^2 \setminus (\mathbb{S}^2 \cap \mathbb{T}^2)$ [Neumann conditions on $\{x_1 = 0\}$, Dirichlet conditions elsewhere].*

Recall first the variational characterization of shrinkers, as critical points of the functional

$$(2.34) \quad A(\Sigma) = \int_{\Sigma} e^{-|X(p)|^2/4} dA(p).$$

As exploited in [KKMØ] in the planar case, the conjugation identity connecting the stability operator in the Gaussian density to the linearized operator in (2.33) can be useful for the analysis. The general identity on any surface in \mathbb{R}^3 is stated in the following lemma.

Lemma 2.8. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a smooth self-shrinker. Then the following conjugation identity holds:*

$$(2.35) \quad \mathcal{L} = e^{\frac{|X|^2}{8}} \left(\Delta + |A|^2 - \frac{|X|^2 + |X^\perp|^2}{16} + 1 \right) e^{\frac{-|X|^2}{8}}.$$

Proof of Lemma 2.8. By an elementary computation using Appendix C in [KKMØ] and noting that $|X^\perp| = 2|H|$. \square

We let $\omega = (x')^2 + (y')^2$ and express functions on the surface of rotation in coordinates as $v = v(t, \theta)$ and get for the intrinsic Laplacian that

$$\Delta = \frac{1}{y\sqrt{\omega}} \frac{\partial}{\partial t} \left(\frac{y}{\sqrt{\omega}} \frac{\partial}{\partial t} \right) + \frac{1}{y^2} \frac{\partial^2}{\partial \theta^2}.$$

The square of the second fundamental form is

$$(2.36) \quad |A|^2 = \frac{1}{\omega^3} \left[(x''y' - y''x')^2 + \frac{(x')^2 \omega^2}{y^2} \right]$$

$$(2.37) \quad = \frac{1}{\omega} \left[\left(\frac{yx' - xy'}{2} - \frac{x'}{y} \right)^2 + \frac{(x')^2}{y^2} \right],$$

Recall that

$$K_{e^{2w}g_0} = e^{-2\omega} (-\Delta_{g_0}\omega + K_{g_0}),$$

so that the Gauß curvature of Angenent's metric is

$$K_{\text{Ang}} = \frac{e^{\frac{x^2+y^2}{2}}}{y^2} \left(1 + \frac{1}{y^2} \right)$$

and the remaining terms give

$$-\frac{1}{2}X \cdot \nabla v + \frac{1}{2}v = -\frac{1}{2} \frac{xx' + yy'}{\omega} \frac{\partial v}{\partial t} + \frac{1}{2}v.$$

By virtue of rotational symmetry, the equation $\mathcal{L}v = 0$ separates, and we expand v by its Fourier series on each radial circle:

$$v(t, \theta) = \sum v_m(t) e^{im\theta},$$

and thus the equations we study are

$$(2.38) \quad \mathcal{L}_m v_m = 0,$$

for the appropriate boundary conditions (e.g. Dirichlet or Neumann), where

$$(2.39) \quad \begin{aligned} \mathcal{L}_m v_m &= \frac{1}{y\sqrt{\omega}} \frac{\partial}{\partial t} \left(\frac{y}{\sqrt{\omega}} \frac{\partial}{\partial t} \right) v_m - \frac{1}{2} \frac{xx' + yy'}{\omega} v'_m \\ &\quad + \left(|A|^2 + \frac{1}{2} - \frac{m^2}{y^2} \right) v_m. \end{aligned}$$

We get thus the following proposition:

Proposition 2.9. *On the compact surface of revolution S_γ with boundary generated by the curve γ , we let*

$$(2.40) \quad M_0(\gamma) := \sup_{p \in \gamma} \left[y(p) \sqrt{\frac{1}{2} + |A(p)|^2} \right].$$

Then the unique solutions to the above Sturm-Liouville problems are:

$$(2.41) \quad v_m \equiv 0, \quad \text{for all } m \geq M_0(\gamma).$$

Proof. This follows immediately from the usual maximum principle. □

Let us instead rewrite the $m = 0$ equation in terms of $\tilde{v}_0 = \sqrt{\omega}v_0$, where we obtain

$$(2.42) \quad \tilde{v}_0'' + K_{\text{Ang}} \tilde{v}_0 = \frac{1}{\sqrt{\omega}} \mathcal{L}_0 v_0 = 0,$$

which is the well-known Jacobi equation for the geodesic γ . Since $\sqrt{\omega} > 0$, Dirichlet conditions for v_0 correspond exactly to Dirichlet conditions for \tilde{v}_0 . Note that since, by symmetry,

$$(2.43) \quad \frac{\partial \sqrt{\omega}}{\partial t}(t_0) = 0, \quad \text{when } \gamma(t_0) \in \{x_1 = 0\},$$

imposing Neumann conditions on $\{x_1 = 0\}$ is also equivalent for v_0 and \tilde{v}_0 .

Lemma 2.10. *For graphs of the form $(x_1, u(x_1))$ the operator \mathcal{L}_0 specializes to $\omega = 1 + (u')^2$, and*

$$\begin{aligned} \omega \mathcal{L}_0 v &= v'' - \frac{x_1}{2} (1 + (u')^2) v' + \left[\left(\frac{u - x_1 u'}{2} - \frac{1}{u} \right)^2 + \frac{1}{u^2} + \frac{1 + (u')^2}{2} \right] v, \\ &=: v'' + P(x_1, u, u') v' + Q(x_1, u, u') v, \end{aligned}$$

when u is a solution to the shrinker equation.

For solution graphs of the form $(f(x_2), x_2)$ we have $\omega = 1 + (f')^2$ and the formula is:

$$\begin{aligned} \omega \mathcal{L}_0 g &= g'' + \left[\frac{1}{x_2} - \frac{x_2}{2} \right] (1 + (f')^2) g' + \left[\left(\frac{x_2 f' - f}{2} - \frac{f'}{x_2} \right)^2 + \frac{(f')^2}{x_2^2} + \frac{1 + (f')^2}{2} \right] g \\ &=: g'' + R(x_2, f, f') g' + S(x_2, f, f') g. \end{aligned}$$

Let us as a preliminary consideration note that the mean curvature H has rotational symmetry, and is an eigenfunction with eigenvalue 1 (see [CM2]):

$$(2.44) \quad \mathcal{L}H = H,$$

and Neumann conditions on $\{x_1 = 0\}$. The profile of \mathbb{T}^2 is convex as shown in [KMØ], and thus since the sign of the mean curvature changes only at points of tangential contact with a straight line from the origin, we see that on \mathbb{T}^2 the function $H \circ \gamma$ has exactly one zero.

By Sturm-Liouville theory, we now conclude from (2.44) that a solution to the Neumann problem for \mathcal{L}_0 , if it exists, needs to have at least two zeros in the interval. However, it of course turns out there are no such non-trivial fields with Neumann conditions, which is what we now will apply more detailed analysis to show.

2.2.1. Surfaces contained in \mathbb{S}^2 . For surfaces contained in \mathbb{S}^2 it is of natural convenience to use polar coordinates. Recall that a curve (ρ, φ) , $\varphi = \arctan(x/z)$ in the xz -plane generated by a function $\rho(\varphi)$ that generates a smooth solution to the self-shrinker equation (2.32) satisfies:

$$(2.45) \quad \rho''(\varphi) = \frac{1}{\rho} \left\{ \rho^2 + 2(\rho')^2 + \left[1 - \frac{\rho^2}{2} - \frac{\rho'}{\rho \tan \varphi} \right] (\rho^2 + (\rho')^2) \right\}.$$

A function w giving a (via a unit normal w.r.t. Euclidean length) variation field, must thus on \mathbb{S}^2 satisfy the equation (see Appendix A in [KKMØ])

$$(2.46) \quad w'' + \frac{1}{\tan \varphi} w' + 4w = 0,$$

with appropriate boundary conditions. The substitution $x = \cos(\varphi)$ in (2.46) gives Legendre's differential equation, and the solution is:

$$w(\varphi) = C_1 P_{l_0}(\cos \varphi) + C_2 Q_{l_0}(\cos \varphi),$$

where P_l and Q_l are respectively the Legendre functions of the first and second kind, and $l_0 = (\sqrt{17} - 1)/2$ is the positive root of $l(l+1) = 4$.

For the surface $S_1 \subseteq \mathbb{S}^2$, which is generated by rotation of the radius 2 quarter-circle, the boundary conditions in the theorem are $w'(0) = 0$ and $w'(\frac{\pi}{2}) = 0$. But since $Q_{l_0}(\cos \varphi)$ has a pole at $\varphi = 0$, we see that $C_2 = 0$. Thus, if we normalize w so that $C_1 = 1$, we have by the first condition that $w = P_{l_0}(\cos \varphi)$. However,

$$(2.47) \quad \frac{dP_{l_0}(\cos \varphi)}{d\varphi} \Big|_{\varphi=\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2} \frac{\sqrt{17} + 1}{\Gamma\left(\frac{1-\sqrt{17}}{4}\right) \Gamma\left(\frac{5+\sqrt{17}}{4}\right)} \neq 0,$$

and hence there is no such Neumann mode. By the preceding, we therefore conclude that under N -fold symmetry, for a large enough $N > 0$,

$$(2.48) \quad \ker \mathcal{L}_{S_1} = \{0\}, \quad \text{on } S_1 = \mathbb{S}^2 \cap \{x_1 \geq 0\} \quad [\text{Neumann conditions}].$$

As for the surface $S_3 \subseteq \mathbb{S}^2$, which we let for definiteness be the component such that $(2, 0) \in S_3$, again $C_1 = 1$ and $C_2 = 0$. Now, since by (2.3)

$$(2.49) \quad P_{l_0}(\cos(\angle(\vec{e}_1, p^+))) \geq P_{l_0}(\cos(\frac{10}{11} + 50\varepsilon_{\text{gap}})) > \frac{3}{50} > 0,$$

we conclude again

$$(2.50) \quad \ker \mathcal{L}_{S_3} = \{0\} \quad [\text{Dirichlet conditions}].$$

For $S_4 \subseteq \mathbb{S}^2$, the component with $(0, 2) \in S_4$, we see that with $w'(\frac{\pi}{2}) = 0$ and fixing $w(\frac{\pi}{2}) = 1$,

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} P_{l_0}(\cos(\frac{\pi}{2})) & Q_{l_0}(\cos(\frac{\pi}{2})) \\ \frac{d}{d\varphi} P_{l_0}(\cos(\frac{\pi}{2})) & \frac{d}{d\varphi} Q_{l_0}(\cos(\frac{\pi}{2})) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

such that with these constants

$$w(\angle(\vec{e}_1, p^+)) \geq w(\frac{11}{10}) - 100\varepsilon_{\text{gap}} > \frac{1}{2} > 0.$$

Thus on the surface S_4 with Neumann and Dirichlet conditions as in the statement of the theorem, we also conclude

$$(2.51) \quad \ker \mathcal{L}_{S_4} = \{0\} \quad [\text{Neumann on } \{x_1 = 0\}, \text{ Dirichlet on } S_4 \cap \mathbb{T}^2].$$

2.2.2. *Surfaces contained in \mathbb{T}^2 .* In this section we will show that the Jacobi fields with Neumann conditions $u'_0(0) = 0$ (and normalized to $u_0(0) = 1$), from respectively the top at bottom of the torus \mathbb{T}^2 from Section 2 have the following end point values at point where \mathbb{T}^2 intersects the round 2-sphere of radius 2:

$$(2.52) \quad \begin{pmatrix} u_0(t_{\text{top}}) \\ u'_0(t_{\text{top}}) \end{pmatrix} = \frac{1}{50} \begin{pmatrix} -22 \pm \varepsilon_{\text{gap}} \\ -37 \pm \varepsilon_{\text{gap}} \end{pmatrix}$$

and

$$(2.53) \quad \begin{pmatrix} u_0(t_{\text{bot}}) \\ u'_0(t_{\text{bot}}) \end{pmatrix} = \frac{1}{50} \begin{pmatrix} -77 \pm \varepsilon_{\text{gap}} \\ -84 \pm \varepsilon_{\text{gap}} \end{pmatrix}$$

This means firstly that the Dirichlet problems on each part (w/ Neumann conditions on \mathcal{P}) have trivial kernel, that is:

$$(2.54) \quad \ker \mathcal{L}_{S_5} = \{0\} \quad [\text{Neumann on } \{x_1 = 0\}, \text{ Dirichlet on } S_5 \cap \mathbb{S}^2],$$

$$(2.55) \quad \ker \mathcal{L}_{S_6} = \{0\} \quad [\text{Neumann on } \{x_1 = 0\}, \text{ Dirichlet on } S_6 \cap \mathbb{S}^2].$$

Secondly, note that non-triviality of the kernel of \mathcal{L} on $\mathbb{T}^2 \cap \{x_1 \geq 0\}$ with Neumann conditions has now been reduced to the conditions

$$\begin{aligned} \alpha u_0(t_{\text{top}}) &= \beta u_0(t_{\text{top}}), \\ \alpha u'_0(t_{\text{top}}) &= -\beta u_0(t_{\text{bot}}), \end{aligned}$$

for a non-zero pair (α, β) or in other words, singularity of the matrix \mathcal{N} :

$$(2.56) \quad \mathcal{N} := \begin{pmatrix} u_0(t_{\text{top}}) & -u_0(t_{\text{bot}}) \\ u'_0(t_{\text{top}}) & u'_0(t_{\text{bot}}) \end{pmatrix}.$$

But in fact from (2.52)–(2.53) we see

$$(2.57) \quad \det \mathcal{N} \geq \frac{9}{5} > 0,$$

so we finally conclude that also

$$\ker \mathcal{L}_{S_2} = \{0\} \quad [\text{Neumann on } \{x_1 = 0\}].$$

To show the required estimates of the Jacobi fields, we consider first an approximate solution V to the linearized equation on the approximate curve Γ from the previous section.

Top: x_1 -graph (Jacobi Equation)

Let us consider the first part of Γ , graphical over the x_1 -axis, on $x_1 \in [0, \frac{3}{5}]$. Here, we have the expressions:

$$\begin{aligned} |\partial_2 P(x_1, \xi, U')| &= 0, \\ |\partial_3 P(x_1, u, \xi')| &= |x_1 \xi'|, \\ |\partial_2 Q(x_1, \xi, U')| &= 2 \left| \left(\frac{\xi - x_1 U'}{2} - \frac{1}{\xi} \right) \left(\frac{1}{2} + \frac{1}{\xi^2} \right) - \frac{1}{\xi^3} \right|, \\ |\partial_3 Q(x_1, u, \xi')| &= \left| x_1 \left(\frac{u - x_1 \xi'}{2} - \frac{1}{u} \right) - \xi' \right|. \end{aligned}$$

Assume for a small $\delta_1^T > 0$ the uniform bounds:

$$(2.58) \quad |V'' + P(x_1, U, U')V' + Q(x_1, U, U')V| \leq \delta_1^T,$$

$$(2.59) \quad |\partial_3 P(x_1, u, \xi')| |v'| + |\partial_3 Q(x_1, u, \xi')| |v| \leq x_1(2x_1^2 + 3),$$

$$(2.60) \quad |\partial_2 Q(x_1, \xi, U')| |v| \leq \frac{8}{5} - \frac{3}{2}x_1^2.$$

We let $\Phi(x_1) := |V'(x_1) - v'(x_1)|$ and estimate:

$$\begin{aligned} \Phi'(x_1) &\leq ||V' - v'|' \stackrel{\text{a.e.}}{=} |V'' - v''| \\ &\leq |P(x_1, U, U')V' - P(x_1, u, u')v'| + |Q(x_1, U, U')V - Q(x_1, u, u')v| + \delta_1^T \\ &\leq |P(x_1, U, U')| |V' - v'| + |P(x_1, U, U') - P(x_1, u, u')| |v'| \\ &\quad + |Q(x_1, U, U')| |V - v| + |Q(x_1, U, U') - Q(x_1, u, u')| |v| + \delta_1^T \\ &\leq |P(x_1, U, U')| |V' - v'| + |Q(x_1, U, U')| |V - v| \\ &\quad + (|\partial_3 P(x_1, u, \xi')| |v'| + |\partial_3 Q(x_1, u, \xi')| |v|) |U' - u'| \\ &\quad + (|\partial_2 P(x_1, \xi, U')| |v'| + |\partial_2 Q(x_1, \xi, U')| |v|) |U - u| + \delta_1^T \\ &\leq |P(x_1, U, U')| \Phi(x_1) + |Q(x_1, U, U')| \int_0^{x_1} \Phi(s) ds + |Q(x_1, U, U')| |V(0) - v(0)| \\ &\quad + x_1(2x_1^2 + 3) |U'(x_1) - u'(x_1)| + (\frac{8}{5} - \frac{3}{2}x_1^2) |U(x_1) - u(x_1)| + \delta_1^T. \end{aligned}$$

We integrate on $[0, x_1]$ to get:

$$\begin{aligned} \Phi(x_1) &\leq \int_0^{x_1} \left[|P(s, U, U')| + \int_0^{x_1} |Q(t, U, U')| dt \right] \Phi(s) ds + \left(\int_0^{x_1} |Q(s, U, U')| ds \right) |V(0) - v(0)| \\ &\quad + \int_0^{x_1} s(2s^2 + 3) |U'(s) - u'(s)| ds + \int_0^{x_1} (\frac{8}{5} - \frac{3}{2}s^2) |U(s) - u(s)| ds + \delta_1^T x_1. \end{aligned}$$

Recall the estimates for $|U'(s) - u'(s)|$, which lead to:

$$\begin{aligned} \int_0^{x_1} s(2s^2 + 3) |U'(s) - u'(s)| ds &\leq [\frac{4}{5} |U(0) - u(0)| + \varepsilon_1^T] \int_0^{x_1} s^2(2s^2 + 3) \exp \left\{ \frac{52}{15} s^2 \right\} ds \\ &\leq \frac{9}{20} |U(0) - u(0)| + \frac{14}{25} \varepsilon_1^T. \end{aligned}$$

Note also that from the estimates for $|U - u|$ from $|U' - u'|$, we have already once estimated the integral of the latter. We now need the sizes of these elementary Gaussian double integrals:

$$\begin{aligned} \int_0^{\frac{3}{5}} (\frac{8}{5} - \frac{3}{2}x_1^2) \left(1 + \int_0^{x_1} \frac{4s}{5} \exp \left\{ \frac{52}{15} s^2 \right\} ds \right) dx_1 &\leq \frac{23}{25}, \\ \int_0^{\frac{3}{5}} (\frac{8}{5} - \frac{3}{2}x_1^2) \int_0^{x_1} s \exp \left\{ \frac{52}{15} s^2 \right\} ds dx_1 &\leq \frac{7}{100}. \end{aligned}$$

Thus

$$\int_0^{\frac{3}{5}} (\frac{8}{5} - \frac{3}{2}s^2) |U(s) - u(s)| ds \leq \frac{23}{25} |U(0) - u(0)| + \frac{7}{100} \varepsilon_1^T.$$

Now, we will furthermore assume the following bounds on the test functions, pertaining to the approximation by ε -geodesics:

$$(2.61) \quad \int_0^{x_1} |P(s, U, U')| ds \leq \int_0^{x_1} s(\frac{7}{5}s^2 + \frac{7}{10}) ds \leq \frac{7}{20}x_1^2(x_1^2 + 1),$$

$$(2.62) \quad \int_0^{x_1} |Q(s, U, U')| ds \leq \int_0^{x_1} (\frac{16}{5}s^2 + \frac{5}{2}) ds \leq \frac{16}{15}x_1^3 + \frac{5}{2}x_1.$$

Applying Grönwall-Bellman again, to these new estimates, we see, using also that $V(0) = v(0) = 1$ by assumption:

$$\begin{aligned} \Phi(x_1) &\leq \left[\int_0^{x_1} s(2s^2 + 3)|U'(s) - u'(s)| ds + \int_0^{x_1} (\frac{8}{5} - \frac{3}{2}s^2)|U(s) - u(s)| ds + \delta_1^T x_1 \right] \times \\ &\quad \exp \left\{ \int_0^{x_1} |P(s, U, U')| ds + x_1 \int_0^{x_1} |Q(s, U, U')| ds \right\} \\ &\leq \left[\frac{137}{100}|U(0) - u(0)| + \frac{63}{100}\varepsilon_1^T + \frac{3}{5}\delta_1^T \right] \times \\ &\quad \exp \left\{ \frac{7}{20}x_1^2(x_1^2 + 1) + \frac{16}{15}x_1^4 + \frac{5}{2}x_1^2 \right\}. \end{aligned}$$

Thus

$$\begin{aligned} |V'(\frac{3}{5}) - v'(\frac{3}{5})| &\leq \frac{23}{5}|U(0) - u(0)| + \frac{11}{5}\varepsilon_1^T + \frac{51}{25}\delta_1^T, \\ |V(\frac{3}{5}) - v(\frac{3}{5})| &\leq \frac{13}{10}|U(0) - u(0)| + \frac{3}{5}\varepsilon_1^T + \frac{11}{50}\delta_1^T. \end{aligned}$$

Here, the last estimate followed by integrating the estimates above, and using again that $V(0) = v(0)$.

Top: x_2 -graph to the sphere (Jacobi Equation)

We consider the next part of Γ , graphical over the x_2 -axis over approximately the region $[y_{\mathbb{S}^2}, u_T(3/5)]$ (in the backwards direction). Here, we have the expressions:

$$\begin{aligned} |\partial_2 R(x_2, \xi, F')| &= 0, \\ |\partial_3 R(x_2, f, \xi')| &= 2 \left| \frac{1}{x_2} - \frac{x_2}{2} \right| |\xi'|, \\ |\partial_2 S(x_2, \xi, F')| &= \left| \left(\frac{1}{x_2} - \frac{x_2}{2} \right) F' + \frac{\xi}{2} \right|, \\ |\partial_3 S(x_2, f, \xi')| &= \left| \left(\frac{1}{x_2} - \frac{x_2}{2} \right) (x_2 \xi' - f) + \frac{4\xi'}{x_2^2} \right|. \end{aligned}$$

Assume for a small $\delta_2^T > 0$ the uniform bounds:

$$(2.63) \quad |G'' + R(x_2, F, F')G' + S(x_2, F, F')G| \leq \delta_2^T,$$

$$(2.64) \quad |\partial_3 R(x_2, u, \xi')| |g'| + |\partial_3 S(x_2, f, \xi')| |g| \leq \frac{3}{10} + \frac{33}{20} (x_2 - y_{\mathbb{S}^2})^5$$

$$(2.65) \quad |\partial_2 S(x_2, \xi, U')| |g| \leq \eta(x_2),$$

where

$$(2.66) \quad \eta(x_2) = \begin{cases} \frac{41}{200} - \frac{3}{10} (x_2 - y_{\mathbb{S}^2})^2, & x_2 \in [y_{\mathbb{S}^2}, \frac{5}{2}], \\ \frac{3}{4} - \frac{5}{2} (x_2 - y_{\mathbb{S}^2}) + \frac{21}{10} (x_2 - y_{\mathbb{S}^2})^2, & x_2 \in [\frac{5}{2}, u_T(3/5)]. \end{cases}$$

We let $\Psi(x_2) := |G'(x_2) - g'(x_2)|$ and estimate:

$$\begin{aligned}
\Psi'(x_2) &\leq ||G' - g'|' \stackrel{\text{a.e.}}{=} |G'' - g''| \\
&\leq |R(x_2, F, F')G' - R(x_2, f, f')g'| + |S(x_2, F, F')G - S(x_2, f, f')g| + \delta_2^T \\
&\leq |R(x_2, F, F')| |G' - g'| + |R(x_2, F, F') - R(x_2, f, f')| |g'| \\
&\quad + |S(x_2, F, F')| |G - g| + |S(x_2, F, F') - S(x_2, f, f')| |g| + \delta_2^T \\
&\leq |R(x_2, F, F')| |G' - g'| + |S(x_2, F, F')| |G - g| \\
&\quad + (|\partial_3 R(x_2, f, \xi')| |g'| + |\partial_3 S(x_2, f, \xi')| |g|) |F' - f'| \\
&\quad + |\partial_2 S(x_2, \xi, F')| |g| |F - f| + \delta_2^T \\
&\leq |R(x_2, F, F')| \Psi(x_2) + |S(x_2, F, F')| \int_0^{x_2} \Psi(s) ds + |S(x_2, F, F')| |G(u_T(3/5)) - g(u_T(3/5))| \\
&\quad + \left[\frac{3}{10} + \frac{33}{20} (x_2 - y_{\mathbb{S}^2})^5 \right] |F'(x_2) - f'(x_2)| + \eta(x_2) |F(x_2) - f(x_2)| + \delta_2^T.
\end{aligned}$$

We integrate on $[x_2, u_T(3/5)]$ to get:

$$\begin{aligned}
\Psi(x_2) &\leq \int_{x_2}^{u_T(3/5)} \left[|R(s, F, F')| + \int_{x_2}^{u_T(3/5)} |S(t, F, F')| dt \right] \Psi(s) ds \\
&\quad + \left(\int_{x_2}^{u_T(3/5)} |S(s, F, F')| ds \right) |G(u_T(3/5)) - g(u_T(3/5))| \\
&\quad + \int_{x_2}^{u_T(3/5)} \left[\frac{3}{10} + \frac{33}{20} (s - y_{\mathbb{S}^2})^5 \right] |F'(s) - f'(s)| ds + \int_{x_2}^{u_T(3/5)} \eta(s) |F(s) - f(s)| ds \\
&\quad + |G'(u_T(3/5)) - g'(u_T(3/5))| + \delta_2^T (u_T(3/5) - x_2).
\end{aligned}$$

Recall, we have above shown the estimates:

$$\begin{aligned}
|F'(x_2) - f'(x_2)| &\leq \left[\left(\frac{13}{8} - \frac{x_2}{2} \right) |F(u_T(3/5)) - f(u_T(3/5))| + |F'(u_T(3/5)) - f'(u_T(3/5))| \right. \\
&\quad \left. + \varepsilon_2^T (u_T(3/5) - x_2) \right] \exp \left\{ \frac{7}{4} - \frac{9}{10} (x_2 - y_{\mathbb{S}^2})^2 + \left(\frac{13}{8} - \frac{x_2}{2} \right) (u_T(3/5) - x_2) \right\}, \\
|F(u_T(3/5)) - f(u_T(3/5))| &\leq \frac{10}{12} \left(\frac{9}{25} \varepsilon_1^T + \frac{23}{80} |U(0) - u(0)| \right), \\
|F'(u_T(3/5)) - f'(u_T(3/5))| &\leq \left(\frac{10}{12} \right)^2 (\varepsilon_1^T + \frac{4}{5} |U(0) - u(0)|).
\end{aligned}$$

We therefore get the bound:

$$\int_{y_{\mathbb{S}^2}}^{u_T(3/5)} \left[\frac{3}{10} + \frac{33}{20} (s - y_{\mathbb{S}^2})^5 \right] |F'(s) - f'(s)| ds \leq \frac{80}{25} |U(0) - u(0)| + 6\varepsilon_1^T + \frac{27}{10} \varepsilon_2^T.$$

Again, we will need the sizes of some elementary Gaussian double integrals:

$$\begin{aligned} & \int_{y_{\mathbb{S}^2}}^{u_T(3/5)} \eta(x_2) \left(1 + \int_{x_2}^{u_T(3/5)} \left(\frac{13}{8} - \frac{s}{2} \right) \exp \left\{ \frac{7}{4} - \frac{9}{10} (s - y_{\mathbb{S}^2})^2 + \left(\frac{13}{8} - \frac{s}{2} \right) (u_T(3/5) - s) \right\} ds \right) dx_2 \leq \frac{29}{50} \\ & \int_{y_{\mathbb{S}^2}}^{u_T(3/5)} \int_{x_2}^{u_T(3/5)} \eta(x_2) \exp \left\{ \frac{7}{4} - \frac{9}{10} (s - y_{\mathbb{S}^2})^2 + \left(\frac{13}{8} - \frac{s}{2} \right) (u_T(3/5) - s) \right\} ds dx_2 \leq \frac{29}{50}, \\ & \int_{y_{\mathbb{S}^2}}^{u_T(3/5)} \int_{x_2}^{u_T(3/5)} \eta(x_2) (u_T(3/5) - s) \exp \left\{ \frac{7}{4} - \frac{9}{10} (s - y_{\mathbb{S}^2})^2 + \left(\frac{13}{8} - \frac{s}{2} \right) (u_T(3/5) - s) \right\} ds dx_2 \leq \frac{19}{50}. \end{aligned}$$

Thus

$$\int_{y_{\mathbb{S}^2}}^{u_T(3/5)} \eta(x_2) |F(x_2) - f(x_2)| dx_2 \leq \frac{1}{2} |U(0) - u(0)| + \frac{29}{50} \varepsilon_1^T + \frac{19}{50} \varepsilon_2^T.$$

Assume once again bounds for the ε -geodesics:

$$(2.67) \quad \int_{x_2}^{u_T(3/5)} |R(s, F, F')| ds \leq -\frac{16}{25} x_2^2 + \frac{22}{10} x_2 - \frac{18}{25},$$

$$(2.68) \quad \int_{x_2}^{u_T(3/5)} |S(s, F, F')| ds \leq -\frac{27}{100} x_2^2 + \frac{7}{25} x_2 - \frac{19}{10}.$$

By Grönwall-Bellman with the new estimates, we see:

$$\begin{aligned} \Psi(x_2) \leq & \left[\left(\int_{x_2}^{u_T(3/5)} |S(s, F, F')| ds \right) |G(u_T(3/5)) - g(u_T(3/5))| \right. \\ & + \int_{x_2}^{u_T(3/5)} \left[\frac{3}{10} + \frac{33}{20} (x_2 - y_{\mathbb{S}^2})^5 \right] |F'(s) - f'(s)| ds \\ & + \int_{x_2}^{u_T(3/5)} \eta(x_2) |F(s) - f(s)| ds + \delta_2^T (u_T(3/5) - x_2) + |G'(u_T(3/5)) - g'(u_T(3/5))| \Big] \times \\ & \exp \left\{ \int_{x_2}^{u_T(3/5)} |R(s, F, F')| ds + (u_T(3/5) - x_2) \int_{x_2}^{u_T(3/5)} |S(s, F, F')| ds \right\}. \end{aligned}$$

Thus we have as before estimates for $|G'(y_{\mathbb{S}^2}) - g'(y_{\mathbb{S}^2})|$, and by integration for $|G(y_{\mathbb{S}^2}) - g(y_{\mathbb{S}^2})|$.

Bottom: x_1 -graph (Jacobi Equation)

Let us consider the first part of Γ , graphical over the x_1 -axis. Here, we have the expressions:

$$\begin{aligned} |\partial_2 P(x_1, \xi, U')| &= 0, \\ |\partial_3 P(x_1, u, \xi')| &= |x_1 \xi'|, \\ |\partial_2 Q(x_1, \xi, U')| &= 2 \left| \left(\frac{\xi - x_1 U'}{2} - \frac{1}{\xi} \right) \left(\frac{1}{2} + \frac{1}{\xi^2} \right) - \frac{1}{\xi^3} \right|, \\ |\partial_3 Q(x_1, u, \xi')| &= \left| x_1 \left(\frac{u - x_1 \xi'}{2} - \frac{1}{u} \right) - \xi' \right|. \end{aligned}$$

Assume for a small $\delta_1^B > 0$ the uniform bounds:

$$(2.69) \quad |V'' + P(x_1, U, U')V' + Q(x_1, U, U')V| \leq \delta_1^B,$$

$$(2.70) \quad |\partial_3 P(x_1, u, \xi')| |v'| + |\partial_3 Q(x_1, u, \xi')| |v| \leq 2,$$

$$(2.71) \quad |\partial_2 Q(x_1, \xi, U')| |v| \leq 41 - 80x_1.$$

We let $\Phi(x_1) := |V'(x_1) - v'(x_1)|$ and estimate:

$$\begin{aligned} \Phi'(x_1) &\leq ||V' - v'|' \stackrel{\text{a.e}}{=} |V'' - v''| \\ &\leq |P(x_1, U, U')V' - P(x_1, u, u')v'| + |Q(x_1, U, U')V - Q(x_1, u, u')v| + \delta_1^B \\ &\leq |P(x_1, U, U')| |V' - v'| + |P(x_1, U, U') - P(x_1, u, u')| |v'| \\ &\quad + |Q(x_1, U, U')| |V - v| + |Q(x_1, U, U') - Q(x_1, u, u')| |v| + \delta_1^B \\ &\leq |P(x_1, U, U')| |V' - v'| + |Q(x_1, U, U')| |V - v| \\ &\quad + (|\partial_3 P(x_1, u, \xi')| |v'| + |\partial_3 Q(x_1, u, \xi')| |v|) |U' - u'| \\ &\quad + |\partial_2 Q(x_1, \xi, U')| |v| |U - u| + \delta_1^B \end{aligned}$$

Hence:

$$\begin{aligned} &\leq |P(x_1, U, U')| \Phi(x_1) + |Q(x_1, U, U')| \int_0^{x_1} \Phi(s) ds + |Q(x_1, U, U')| |V(0) - v(0)| \\ &\quad + 2|U'(x_1) - u'(x_1)| + (41 - 80x_1)|U(x_1) - u(x_1)| + \delta_1^B. \end{aligned}$$

We integrate on $[0, x_1]$ to get:

$$\begin{aligned} \Phi(x_1) &\leq \int_0^{x_1} \left[|P(s, U, U')| + \int_0^{x_1} |Q(t, U, U')| dt \right] \Phi(s) ds + \left(\int_0^{x_1} |Q(s, U, U')| ds \right) |V(0) - v(0)| \\ &\quad + 2 \int_0^{x_1} |U'(s) - u'(s)| ds + \int_0^{x_1} (41 - 80s) |U(s) - u(s)| ds + \delta_1^B x_1. \end{aligned}$$

Recall the estimates:

$$\begin{aligned}
2 \int_0^{x_1} |U'(s) - u'(s)| ds &\leq 2|U(0) - u(0)| \int_0^{x_1} \left(\frac{29}{5}s + \frac{1}{20}\right) \exp\left\{\frac{49}{5}s^2 + \frac{s}{4}\right\} ds \\
&\quad + 2\varepsilon_1^B \int_0^{x_1} s \exp\left\{\frac{49}{5}s^2 + \frac{s}{4}\right\} ds \\
&\leq \frac{36}{5}|U(0) - u(0)| + \frac{6}{5}\varepsilon_1^B.
\end{aligned}$$

Note also that from the estimates for $|U - u|$ from $|U' - u'|$, we have already once estimated the integral of the latter. We now need the sizes of these elementary Gaussian double integrals:

$$\begin{aligned}
&\int_0^{\frac{1}{2}} (41 - 80x_1) \left(1 + \int_0^{x_1} \left(\frac{29}{5}s + \frac{1}{20}\right) \exp\left\{\frac{49}{5}s^2 + \frac{s}{4}\right\} ds\right) dx_1 \leq \frac{67}{5}, \\
&\int_0^{\frac{1}{2}} \int_0^{x_1} (41 - 80x_1) s \exp\left\{\frac{49}{5}s^2 + \frac{s}{4}\right\} ds dx_1 \leq \frac{12}{25}.
\end{aligned}$$

Thus

$$\int_0^{\frac{1}{2}} (41 - 80x_1) |U(x_1) - u(x_1)| dx_1 \leq \frac{67}{5}|U(0) - u(0)| + \frac{12}{25}\varepsilon_1^B.$$

Now, we will furthermore assume the bounds pertaining to the ε -geodesics:

$$(2.72) \quad \int_0^{x_1} |P(s, U, U')| ds \leq \frac{4}{9}x_1^2,$$

$$(2.73) \quad \int_0^{x_1} |Q(s, U, U')| ds \leq 4 - 16(x_1 - \frac{1}{2})^2.$$

Again, by Grönwall-Bellman we see (with $V(0) = v(0)$):

$$\begin{aligned}
\Phi(x_1) &\leq \left[2 \int_0^{x_1} |U'(s) - u'(s)| ds + \int_0^{x_1} (41 - 80s) |U(s) - u(s)| ds + \delta_1^B x_1\right] \times \\
&\quad \exp\left\{\int_0^{x_1} |P(s, U, U')| ds + x_1 \int_0^{x_1} |Q(s, U, U')| ds\right\} \\
&\leq \left[\frac{103}{5}|U(0) - u(0)| + \frac{42}{25}\varepsilon_1^B + \delta_1^B x_1\right] \times \\
&\quad \exp\left\{\frac{4}{9}x_1^2 + 4x_1 - 16x_1(x_1 - \frac{1}{2})^2\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
|V'(\frac{1}{2}) - v'(\frac{1}{2})| &\leq 171|U(0) - u(0)| + 14\varepsilon_1^B + \frac{38}{9}\delta_1^B, \\
|V(\frac{1}{2}) - v(\frac{1}{2})| &\leq 31|U(0) - u(0)| + \frac{51}{20}\varepsilon_1^B + \frac{11}{5}\delta_1^B.
\end{aligned}$$

Here, the last estimate followed by integration and using again $V(0) = v(0)$.

Bottom: x_2 -graph to cylinder (Jacobi Equation)

We consider the next part of Γ , graphical over the x_2 -axis over $[a_0, \sqrt{2}]$. Here, we have the expressions:

$$\begin{aligned} |\partial_2 R(x_2, \xi, F')| &= 0, \\ |\partial_3 R(x_2, f, \xi')| &= 2 \left| \frac{1}{x_2} - \frac{x_2}{2} \right| |\xi'|, \\ |\partial_2 S(x_2, \xi, F')| &= \left| \left(\frac{1}{x_2} - \frac{x_2}{2} \right) F' + \frac{\xi}{2} \right|, \\ |\partial_3 S(x_2, f, \xi')| &= \left| \left(\frac{1}{x_2} - \frac{x_2}{2} \right) (x_2 \xi' - f) + \frac{4\xi'}{x_2^2} \right|. \end{aligned}$$

Assume for a small $\delta_2^B > 0$ the uniform bounds:

$$(2.74) \quad |G'' + R(x_2, F, F')G' + S(x_2, F, F')G| \leq \delta_2^B,$$

$$(2.75) \quad |\partial_3 R(x_2, u, \xi')| |g'| + |\partial_3 S(x_2, f, \xi')| |g| \leq \frac{4}{5} + 8 \left(x_2 - \sqrt{2} \right)^2,$$

$$(2.76) \quad |\partial_2 S(x_2, \xi, U')| |g| \leq \frac{24}{50} - \frac{3}{4} \left(x_2 - \sqrt{2} \right)^2.$$

We let $\Psi(x_2) := |G'(x_2) - g'(x_2)|$ and estimate:

$$\begin{aligned} \Psi'(x_2) &\leq ||G' - g'|' \stackrel{\text{a.e}}{=} |G'' - g''| \\ &\leq |R(x_2, F, F')G' - R(x_2, f, f')g'| + |S(x_2, F, F')G - S(x_2, f, f')g| + \delta_2^B \\ &\leq |R(x_2, F, F')| |G' - g'| + |R(x_2, F, F') - R(x_2, f, f')| |g'| \\ &\quad + |S(x_2, F, F')| |G - g| + |S(x_2, F, F') - S(x_2, f, f')| |g| + \delta_2^B \\ &\leq |R(x_2, F, F')| |G' - g'| + |S(x_2, F, F')| |G - g| \\ &\quad + (|\partial_3 R(x_2, f, \xi')| |g'| + |\partial_3 S(x_2, f, \xi')| |g|) |F' - f'| \\ &\quad + |\partial_2 S(x_2, \xi, F')| |g| |F - f| + \delta_2^B \\ &\leq |R(x_2, F, F')| \Psi(x_2) + |S(x_2, F, F')| \int_0^{x_2} \Psi(s) ds + |S(x_2, F, F')| |G(a_0) - g(a_0)| \\ &\quad + \left[\frac{4}{5} + 8 \left(x_2 - \sqrt{2} \right)^2 \right] |F'(x_2) - f'(x_2)| + \left[\frac{24}{50} - \frac{3}{4} \left(x_2 - \sqrt{2} \right)^2 \right] |F(x_2) - f(x_2)| + \delta_2^B. \end{aligned}$$

We integrate on $[a_0, x_2]$ to get:

$$\begin{aligned} \Psi(x_2) &\leq \int_{a_0}^{x_2} \left[|R(s, F, F')| + \int_{a_0}^{x_2} |S(t, F, F')| dt \right] \Psi(s) ds + \left(\int_{a_0}^{x_2} |S(s, F, F')| ds \right) |G(a_0) - g(a_0)| \\ &\quad + \int_{a_0}^{x_2} \left[\frac{4}{5} + 8(s - \sqrt{2})^2 \right] |F'(s) - f'(s)| ds + \int_{a_0}^{x_2} \left[\frac{24}{50} - \frac{3}{4}(s - \sqrt{2})^2 \right] |F(s) - f(s)| ds \\ &\quad + |G'(a_0) - g'(a_0)| + \delta_2^B (x_2 - a_0). \end{aligned}$$

Recall, we have above shown the estimates:

$$\begin{aligned}
|F'(x_2) - f'(x_2)| &\leq \left[\left(\frac{16}{25}x_2 - \frac{3}{7} \right) |F(a_0) - f(a_0)| + |F'(a_0) - f'(a_0)| + \varepsilon_2^B (x_2 - a_0) \right] \\
&\quad \times \exp \left\{ \frac{9}{10} - \frac{11}{10} \left(x_2 - \frac{3}{2} \right)^2 + (x_2 - a_0) \left(\frac{16}{25}x_2 - \frac{3}{7} \right) \right\}, \\
|F(a_0) - f(a_0)| &\leq \frac{10}{12} \left(\frac{3}{5}\varepsilon_1^B + \frac{23}{5}|U(0) - u(0)| \right) \\
|F'(a_0) - f'(a_0)| &\leq \left(\frac{10}{12} \right)^2 \left(\frac{28}{5}\varepsilon_1^B + 39|U(0) - u(0)| \right).
\end{aligned}$$

We therefore get the bound:

$$\int_{a_0}^{\sqrt{2}} \left[\frac{4}{5} + 8(s - \sqrt{2})^2 \right] |F'(s) - f'(s)| ds \leq 78|U(0) - u(0)| + 12\varepsilon_1^B + \frac{9}{10}\varepsilon_2^B.$$

Again, we will need the sizes of some elementary Gaussian double integrals:

$$\begin{aligned}
&\int_{a_0}^{\sqrt{2}} \left[\frac{24}{50} - \frac{3}{4}(x_2 - \sqrt{2})^2 \right] \left(1 + \int_{a_0}^{x_2} \left(\frac{16}{25}s - \frac{3}{7} \right) \exp \left\{ \frac{9}{10} - \frac{11}{10} \left(s - \frac{3}{2} \right)^2 + (s - a_0) \left(\frac{16}{25}s - \frac{3}{7} \right) \right\} ds \right) dx_2 \\
&\leq \frac{3}{10}, \\
&\int_{a_0}^{\sqrt{2}} \int_{a_0}^{x_2} \left[\frac{24}{50} - \frac{3}{4}(x_2 - \sqrt{2})^2 \right] \exp \left\{ \frac{9}{10} - \frac{11}{10} \left(x_2 - \frac{3}{2} \right)^2 + (x_2 - a_0) \left(\frac{16}{25}x_2 - \frac{3}{7} \right) \right\} ds dx_2 \leq \frac{1}{5}, \\
&\int_{a_0}^{\sqrt{2}} \int_{a_0}^{x_2} \left[\frac{24}{50} - \frac{3}{4}(x_2 - \sqrt{2})^2 \right] (s - a_0) \exp \left\{ \frac{9}{10} - \frac{11}{10} \left(x_2 - \frac{3}{2} \right)^2 + (x_2 - a_0) \left(\frac{16}{25}x_2 - \frac{3}{7} \right) \right\} ds dx_2 \leq \frac{7}{125}.
\end{aligned}$$

Thus

$$\int_{a_0}^{\sqrt{2}} \left[\frac{24}{50} - \frac{3}{4}(x_2 - \sqrt{2})^2 \right] |F(x_2) - f(x_2)| dx_2 \leq \frac{197}{30}|U(0) - u(0)| + \frac{47}{50}\varepsilon_1^B + \frac{7}{125}\varepsilon_2^B.$$

Assume again bounds for the ε -geodesics:

$$(2.77) \quad \int_{a_0}^{x_2} |R(s, F, F')| ds \leq \frac{9}{20} - \frac{3}{4}(x_2 - \sqrt{2})^2,$$

$$(2.78) \quad \int_{a_0}^{x_2} |S(s, F, F')| ds \leq x_2 - \frac{2}{5}.$$

By Grönwall-Bellman with the new estimates, we see:

$$\begin{aligned}
\Psi(x_2) &\leq \left[\left(\int_{a_0}^{x_2} |S(s, F, F')| ds \right) |G(a_0) - g(a_0)| + \int_{a_0}^{x_2} \left[\frac{4}{5} + 8(s - \sqrt{2})^2 \right] |F'(s) - f'(s)| ds \right. \\
&\quad \left. + \int_{a_0}^{x_2} \left[\frac{24}{50} - \frac{3}{4}(x_2 - \sqrt{2})^2 \right] |F(s) - f(s)| ds + \delta_2^B(x_2 - a_0) + |G'(a_0) - g'(a_0)| \right] \times \\
&\quad \exp \left\{ \int_{a_0}^{x_2} |R(s, F, F')| ds + (x_2 - a_0) \int_{a_0}^{x_2} |S(s, F, F')| ds \right\} \\
&\leq \frac{63}{20} \left[31|U(0) - u(0)| + \frac{51}{20}\varepsilon_1^B + \frac{11}{5}\delta_1^B + 78|U(0) - u(0)| + 12\varepsilon_1^B + \frac{9}{10}\varepsilon_2^B + \frac{197}{30}|U(0) - u(0)| \right. \\
&\quad \left. + \frac{47}{50}\varepsilon_1^B + \frac{7}{125}\varepsilon_2^B + (\sqrt{2} - a_0)\delta_2^B + \frac{171|U(0) - u(0)| + 14\varepsilon_1^B + \frac{38}{9}\delta_1^B}{|u'(\frac{1}{2})|} \right]
\end{aligned}$$

Thus

$$\begin{aligned}
|G'(\tfrac{1}{2}) - v'(\tfrac{1}{2})| &\leq 813|U(0) - u(0)| + 86\varepsilon_1^B + 3\varepsilon_2^B + \frac{95}{27}\delta_1^B + (\sqrt{2} - a_0)\delta_2^B, \\
|G(\tfrac{1}{2}) - v(\tfrac{1}{2})| &\leq 522|U(0) - u(0)| + 56\varepsilon_1^B + 2\varepsilon_2^B + 11\delta_1^B + \frac{9}{10}\delta_2^B.
\end{aligned}$$

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